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Choosing k from m : Feasible elimination procedures reconsidered [☆]



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ABSTRACT

We show that feasible elimination procedures (Peleg, 1978) can be used to select k from m alternatives. An important advantage of this method is the core property: no coalition can guarantee an outcome that is preferred by all its members. We also show that the problem of determining whether a specific k -tuple can result from a feasible elimination procedure is computationally equivalent to the problem of finding a maximal matching in a bipartite graph.

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1. Introduction

This paper focuses on the question of how to choose a set of k alternatives from a set of $m > k$ alternatives, by aggregating the preferences of voters over alternatives in a way that cannot be manipulated in the following sense: no coalition of voters, by possibly misrepresenting their preferences, can guarantee an outcome that all members of the coalition prefer to the outcome obtained by sincere, truthful voting. For instance, the set of k alternatives is a department board, which has to be composed from a set of available candidates, and we are looking for a voting method such that no coalition of voters, by not voting sincerely, can guarantee a board that all members of the coalition prefer. In order to achieve this, we will use an extension of a method that was originally proposed by Peleg (1978) as a reaction to the Gibbard–Satterthwaite Theorem.

The Gibbard–Satterthwaite Theorem (Gibbard, 1973, and Satterthwaite, 1975) states that every non-dictatorial social choice function whose range contains at least three alternatives, is manipulable: there exists a profile of preferences and a voter who, by deviating and reporting a preference different from his true one, can obtain an outcome which he prefers, according to his true preference, to the sincere outcome. From a game-theoretic point of view, a social choice function combined with a (true) preference profile is a game in strategic form, the strategy set of each player being the set of preferences over the alternatives; outcomes of the game are evaluated according to the true preferences. Peleg (1978) showed that there are reasonable (anonymous, Maskin monotonic) social choice functions such that the resulting game has a strong Nash equilibrium resulting in the sincere outcome whatever the true preferences. Social choice functions with this

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property are called exactly and strongly consistent (ESC). In particular, for an ESC social choice function the sincere outcome is in the core of the relevant voting game: this means that there is no coalition of voters that has a strategy profile which guarantees an alternative that all members of the coalition prefer over the sincere outcome. Of course, the sincere outcome may not be the unique outcome of a strong Nash equilibrium.¹

The central tool, introduced in Peleg (1978), to obtain these social choice functions is the concept of a feasible elimination procedure. In such a procedure, applied to a profile of preferences, alternatives are eliminated one by one, until a final alternative remains: this is called a maximal alternative. In this paper we use this procedure to select $k > 1$ alternatives, simply by taking the last k instead of only the last alternative. We show that for at least two extensions of voters' preferences over the alternatives to (ordered) k -tuples of alternatives, this method has the core property: no coalition can guarantee an outcome (k -tuple) that is preferred by all its members. Formally, this core is defined as the core of the effectivity function induced by this method. We show, by an example, that some well-known existing methods (single transferable vote, plurality, plurality with run-off) violate the core property.

We also show that the problem of determining whether a specific k -tuple can result from a feasible elimination procedure is computationally equivalent to the problem of finding a maximal matching in a bipartite graph. The latter problem can be solved in polynomial time (Hopcroft and Karp, 1973).

Section 2 introduces feasible elimination procedures and Section 3 shows that these procedures provide a method to choose k alternatives from a set of m alternatives such that no coalition of voters can guarantee a better result by manipulation. We also show that this method is Maskin monotonic. Section 4 considers the computational aspect, and Section 5 concludes.

Notations. The following basic notations are used throughout. For a set D , $|D|$ denotes the cardinality of D , $P(D)$ the power set, i.e., the set of all subsets of D , and $P_0(D)$ the set of all nonempty subsets of D .

2. Preliminaries

Let A be a set of m alternatives, $m \geq 2$, and let $N = \{1, \dots, n\}$, $n \geq 2$, be a set of voters. Denote by L the set of all linear orderings, i.e., complete, antisymmetric and transitive binary relations, of A . An element of L is also called a *preference*, and an element of L^N a *preference profile*. If $xR^i y$ for some $x, y \in A$ and $i \in N$, where $R^i \in L$, then this is interpreted as voter i strictly preferring x over y .

A *social choice correspondence* (SCC) is a function $H : L^N \rightarrow P_0(A)$. An SCC H is *anonymous* if for all $R^N \in L^N$ and for all permutations π of N , $H(R^1, \dots, R^n) = H(R^{\pi(1)}, \dots, R^{\pi(n)})$. It is *Paretian* if for all $x, y \in A$ and $R^N \in L^N$, if $x \neq y$ and $yR^i x$ for all $i \in N$, then $x \notin H(R^N)$. It is *Maskin monotonic* (Maskin, 1999) if it satisfies the following. Let $R^N, Q^N \in L^N$ and let $x \in H(Q^N)$. If $xQ^i y$ implies $xR^i y$ for all $y \in A$ and $i \in N$, then $x \in H(R^N)$.

In this paper we are especially interested in the SCC derived from feasible elimination procedures, introduced by Peleg (1978).

Definition 2.1. Assume that $n + 1 \geq m$ and let $\beta : A \rightarrow \{1, 2, \dots\}$ satisfy $\sum_{x \in A} \beta(x) = n + 1$. Let $R^N \in L^N$. A *feasible elimination procedure* (f.e.p.) for R^N is a sequence $(x_1, C_1; \dots; x_{m-1}, C_{m-1}; x_m)$ such that

- 1) $A = \{x_1, \dots, x_m\}$,
- 2) C_1, \dots, C_{m-1} are pairwise disjoint subsets of N and $|C_j| = \beta(x_j)$ for $j = 1, \dots, m - 1$,
- 3) For all $j = 1, \dots, m - 1$, $x_k R^i x_j$ for all $k = j + 1, \dots, m$ and $i \in C_j$.

Thus, in a feasible elimination procedure, we can consecutively eliminate bottom alternatives x_1, x_2, \dots, x_{m-1} of the preference profile; when we eliminate an alternative x_j we also eliminate the preferences of $\beta(x_j)$ voters who have x_j at bottom in the current profile. From the assumptions in the definition it easily follows that there exists always at least one f.e.p. Henceforth in this paper we assume $n + 1 \geq m$. An alternative y is R^N -*maximal* if there exists an f.e.p. $(x_1, C_1; \dots; x_{m-1}, C_{m-1}; y)$. We denote

$$M(R^N) = \{x \in A : x \text{ is } R^N\text{-maximal}\}.$$

It is not difficult to see that M is an anonymous and Paretian social choice correspondence.² Also, M is Maskin monotonic. This follows from Lemma 3.7 below.

¹ Peleg (1978) has been followed by several investigations of the set of exactly and strongly consistent social choice functions: Dutta and Pattanaik (1978), Polishchuk (1978), Ishikawa and Nakamura (1980), Oren (1981), Kim and Roush (1981), Holzman (1986), and Peleg and Peters (2006). Also the books of Peleg (1984), Abdou and Keiding (1991), Danilov and Sotskov (2002), and Peleg and Peters (2010) contain lengthy analytic discussions of consistent voting systems.

² Clearly, M depends on β , but this is suppressed from notation if no confusion is likely.

Remark 2.2. As mentioned in the Introduction, the interest of feasible elimination procedures and the SCC M lies in the fact that, for every social choice function F (i.e., single-valued SCC) that is a selection from M , the game in which each voter reports a preference and the outcome is determined by F , has a strong Nash equilibrium resulting in the sincere outcome, i.e., the alternative resulting if the players would report truthfully. This provides a way to obtain the sincere outcome in spite of the Gibbard–Satterthwaite result that any reasonable social choice function is manipulable. In fact, it can be shown that under anonymity and a very reasonable additional condition, namely that no single voter can veto an alternative, selecting from maximal alternatives is the only way to achieve this (see Peleg and Peters, 2010).

3. Choosing k from m

In this section we show how the concept of a feasible elimination procedure can be used to select not just one alternative, but an ordered k -tuple of alternatives, where $1 \leq k \leq m - 1$. We show, in particular, that the resulting method has the ‘core property’: there is no subset of voters that can vote strategically in order to guarantee a better outcome for all its members. This is in contrast to some well-known existing methods that can be used for choosing k from m – see Example 3.6.

In order to formalize this we use the concept of an effectivity function, defined here for some general finite set of alternatives Z .

Definition 3.1. A function $E : P(N) \rightarrow P(P_0(Z))$ is an *effectivity function* (EF) if: 1) $E(N) = P_0(Z)$, 2) $Z \in E(S)$ for every $S \in P_0(N)$, and 3) $E(\emptyset) = \emptyset$.

If $B \in E(S)$ for an effectivity function E , then we say that coalition S is *effective* for the set of alternatives B . The interpretation is that S can ‘guarantee’ that the ‘final’ alternative is in B .

Let $k \in \{1, \dots, m - 1\}$ and denote by \bar{A}^k the set

$$\{(x_1, \dots, x_k) \in \bar{A}^k : |\{x_1, \dots, x_k\}| = k\}.$$

As before, assume that $n + 1 \geq m$ and let $\beta : A \rightarrow N$ satisfy $\sum_{x \in A} \beta(x) = n + 1$. For these weights, we define $M^k : L^N \rightarrow P_0(\bar{A}^k)$ by: $(y_1, \dots, y_k) \in M^k(R^N)$ if there is an f.e.p. $(x_1, C_1; \dots; x_{m-1}, C_{m-1}; x_m)$ such that $(y_1, \dots, y_k) = (x_{m-k+1}, \dots, x_m)$. In words, M^k assigns to a preference profile all tuples of k last alternatives in any f.e.p. for that profile. Clearly, $M^1 = M$. Define the function $E^{M^k} : P(N) \rightarrow P(P_0(\bar{A}^k))$ by $E^{M^k}(\emptyset) = \emptyset$ and, for $S \neq \emptyset$ and $B^k \subseteq \bar{A}^k$:

$$B^k \in E^{M^k}(S) \Leftrightarrow \exists R^S \in L^S [\forall R^{N \setminus S} \in L^{N \setminus S} : M^k(R^S, R^{N \setminus S}) \subseteq B^k].$$

Since, in particular, $\{(x_1, \dots, x_k) \in \bar{A}^k\} \in E^{M^k}(N)$ for each $(x_1, \dots, x_k) \in \bar{A}^k$ (namely, for each $i \in N$ let R^i satisfy $x_k R^i x_{k-1} \dots x_2 R^i x_1 R^i \dots$), we have that E^{M^k} is an effectivity function, with $Z = \bar{A}^k$ in Definition 3.1.

In this paper we consider two kinds of preferences of the voters for k -tuples. Both kinds of preferences extend the original preferences in L to linear orderings on \bar{A}^k . In the first extension we respect the order of a k -tuple and lexicographically compare alternatives, starting from the rightmost one. Formally we have:

(P1) Let $R \in L$. Then for $\bar{x} = (x_1, \dots, x_k) \in \bar{A}^k$ and $\bar{y} = (y_1, \dots, y_k) \in \bar{A}^k$ we define $\bar{x} \bar{R} \bar{y}$ if $x_\ell R y_\ell$, where $\ell = \max\{j \in \{1, \dots, k\} : x_j \neq y_j\}$.

In the second extension we ignore the order of a k -tuple and lexicographically compare alternatives, starting from the worst one. Formally we have:

(P2) Let $R \in L$. For $\bar{x} = (x_1, \dots, x_k) \in \bar{A}^k$ and $\bar{y} = (y_1, \dots, y_k) \in \bar{A}^k$ reorder the alternatives so that $x_{i_1} R \dots R x_{i_k}$ and $y_{j_1} R \dots R y_{j_k}$. Then we define $\bar{x} \bar{R} \bar{y}$ if $x_{i_\ell} R y_{j_\ell}$, where $\ell = \max\{h \in \{1, \dots, k\} : x_{i_h} \neq y_{j_h}\}$.

Preference extension P1 could apply, for instance, if a president and a vice-president of a society have to be chosen ($k = 2$). Then we first compare the candidates for president and, in case these are equal, we compare the candidates for vice-president. Extension P2 could apply when the members of a board to be chosen are on equal foot. Then we first compare the worst candidates. If these are the same, then we compare second worst candidates, etc.

In what follows we concentrate on these two preference extensions. For another extension and discussion, see Section 5. Before we can formulate the main results of this section we need another definition.

Definition 3.2. Let $E : P(N) \rightarrow P(P_0(Z))$ be an EF, let R^N be a profile of linear orderings over Z , let $S \in P_0(N)$, let $B \in E(S)$, and let $z \in Z \setminus B$. We say that z is *dominated by B via S at R^N* if $z' R^i z$ for all $z' \in B$ and $i \in S$; z is *dominated at R^N* if there exist B and S as above such that B dominates z via S at R^N . The *core of E at R^N* , $C(E, R^N)$, is the set of all alternatives that are not dominated at R^N .

We will now show that M^k selects only core alternatives under both preference extensions (P1) and (P2), starting with the latter.

Theorem 3.3. *Let $R^N \in L^N$, and let \bar{R}^N be the profile of extended preferences according to (P2). Then $M^k(R^N) \subseteq C(E^{M^k}, \bar{R}^N)$.*

Proof. Consider an f.e.p. $f^* = (x_1, C_1; \dots; x_{m-1}, C_{m-1}; x_m)$ for R^N , resulting in $(x_{m-k+1}, \dots, x_m) \in M^k(R^N)$. Suppose there is an $S \subseteq N$ and a set $B^k \subseteq \bar{A}^k$ such that $B^k \in E^{M^k}(S)$, $(x_{m-k+1}, \dots, x_m) \notin B^k$, and $\bar{y} \bar{R}^i(x_{m-k+1}, \dots, x_m)$ for all $i \in S$ and $\bar{y} \in B^k$. We derive a contradiction, which completes the proof of the theorem.

Let $K = \{x_{m-k+1}, \dots, x_m\}$ and let

$$B = \{x \in A : x \in \{y_1, \dots, y_k\} \text{ for some } (y_1, \dots, y_k) \in B^k\}.$$

Note that $B \setminus K \neq \emptyset$, otherwise we would have $B = K$, which is not possible in view of the preferences of the voters in S . Consider any $x_j \in B \setminus K$, and $\bar{y} = (y_1, \dots, y_k) \in B^k$ with $y_\ell = x_j$ for some $\ell \in \{1, \dots, k\}$. Since $\bar{y} \bar{R}^i(x_{m-k+1}, \dots, x_m)$ for all $i \in S$, by (P2) there is for each $i \in S$ an $x_h \in K$ with $x_j R^i x_h$. Since x_j is eliminated according to f^* prior to the alternatives of K , this implies that $C_j \cap S = \emptyset$. Hence, the alternatives of $B \setminus K$ are eliminated according to f^* via only voters in $N \setminus S$. Therefore we have $|N \setminus S| \geq \beta(B \setminus K)$.

Let $Q^S \in L^S$ be a profile resulting in B^k , i.e., $M^k(Q^S, V^{N \setminus S}) \subseteq B^k$ for all $V^{N \setminus S} \in L^{N \setminus S}$. Consider, in particular, a profile $V^{N \setminus S} \in L^{N \setminus S}$ such that each $x_j \in B \setminus K$ is a bottom alternative for at least $\beta(x_j)$ voters in $N \setminus S$ (this is possible since $|N \setminus S| \geq \beta(B \setminus K)$). Then, for this specific profile $(Q^S, V^{N \setminus S})$, there is an f.e.p. in which all alternatives of $B \setminus K$ are eliminated first. This implies that the resulting k -tuple, say \bar{y} , of B^k can only contain alternatives of K , contradicting $\bar{y} \bar{R}^i(x_{m-k+1}, \dots, x_m)$ for all $i \in S$. \square

Theorem 3.4. *Let $R^N \in L^N$, and let \bar{R}^N be the profile of extended preferences according to (P1). Then $M^k(R^N) \subseteq C(E^{M^k}, \bar{R}^N)$.*

Proof. Let f^* , S , B^k , and Q^S be as in the proof of Theorem 3.3. The proof proceeds in several steps.

Step 1 Let

$$B(m) = \{x_j \in A : j \in \{1, \dots, m-1\} \text{ and } x_j = y_k \text{ for some } (y_1, \dots, y_k) \in B^k\}.$$

By (P1), we have $x_j R^i x_m$ for all $x_j \in B(m)$ and $i \in S$. Hence, $C_j \subseteq N \setminus S$ for all j with $x_j \in B(m)$, so that $|N \setminus S| \geq \beta(B(m))$. Consider a profile $V^{N \setminus S}(m) \in L^{N \setminus S}$ where all $x_j \in B(m)$ are ranked at bottom positions for at least $\beta(x_j)$ voters in $N \setminus S$. Let $B^k(m)$ be the subset of B^k , consisting of all k -tuples which are obtained by f.e.p.'s for the profile $(Q^S, V^{N \setminus S}(m))$ such that the alternatives of $B(m)$ are eliminated first, with x_j before x_ℓ whenever $x_j, x_\ell \in B(m)$ and $j < \ell$; clearly, all these f.e.p.'s result in x_m , so that $y_k = x_m$ for all $(y_1, \dots, y_k) \in B^k(m)$.

Step 2 Let

$$B(m-1) = \{x_j : j \in \{1, \dots, m-2\} \text{ and } x_j = y_{k-1} \text{ for some } (y_1, \dots, y_{k-1}, x_m) \in B^k(m)\}.$$

By (P1), we have $x_j R^i x_{m-1}$ for all $x_j \in B(m-1)$ and $i \in S$. Hence, $C_j \subseteq N \setminus S$ for all j with $x_j \in B(m-1)$, so that $|N \setminus S| \geq \beta(B(m) \cup B(m-1))$. Consider a profile $V^{N \setminus S}(m-1) \in L^{N \setminus S}$ where all $x_j \in B(m) \cup B(m-1)$ are ranked at bottom positions for at least $\beta(x_j)$ voters in $N \setminus S$. Let $B^k(m-1)$ be the subset of $B^k(m)$, consisting of all k -tuples which are obtained by f.e.p.'s for the profile $(Q^S, V^{N \setminus S}(m-1))$ such that the alternatives of $B(m) \cup B(m-1)$ are eliminated first, with x_j before x_ℓ whenever $x_j, x_\ell \in B(m) \cup B(m-1)$ and $j < \ell$; clearly, all these f.e.p.'s have (x_{m-1}, x_m) as last pair, so that $y_{k-1} = x_{m-1}$ and $y_k = x_m$ for all $(y_1, \dots, y_k) \in B^k(m-1)$.

\vdots

Step k Let

$$B(m-k+1) = \{x_j : j \in \{1, \dots, m-k\} \text{ and } x_j = y_1 \text{ for some } (y_1, x_{m-k+2}, \dots, x_m) \in B^k(m-k+2)\}.$$

By (P1), we have $x_j R^i x_{m-k+1}$ for all $x_j \in B(m-k+1)$ and $i \in S$. Hence, $C_j \subseteq N \setminus S$ for all j with $x_j \in B(m-k+1)$, so that $|N \setminus S| \geq \beta(B(m) \cup \dots \cup B(m-k+1))$. Consider a profile $V^{N \setminus S}(m-k+1) \in L^{N \setminus S}$ where all $x_j \in B(m) \cup \dots \cup B(m-k+1)$ are ranked at bottom positions for at least $\beta(x_j)$ voters in $N \setminus S$. Let $B^k(m-k+1)$ be the subset of $B^k(m-k+2)$, consisting of all k -tuples which are obtained by f.e.p.'s for the profile $(Q^S, V^{N \setminus S}(m-k+1))$ such that the alternatives of $B(m) \cup \dots \cup$

$B(m - k + 1)$ are eliminated first, with x_j before x_ℓ whenever $x_j, x_\ell \in B(m) \cup \dots \cup B(m - k + 1)$ and $j < \ell$; clearly, all these f.e.p.'s have (x_{m-k+1}, \dots, x_m) as last k -tuple. This, however, implies that $(x_{m-k+1}, \dots, x_m) \in B^k$, which is a contradiction. \square

The following example shows that the set inclusions in [Theorems 3.3 and 3.4](#) can be strict if $k > 1$. (For $k = 1$ we do have $M(R^N) = C(E^M, R^N)$ for all $R^N \in L^N$ by Lemma 9.3.2 and Theorem 9.3.6 in [Peleg and Peters, 2010](#).)

Example 3.5. Let $A = \{x, y, z\}$, $N = \{1, \dots, 4\}$, $\beta(x) = 1$, and $\beta(y) = \beta(z) = 2$. Consider the profile R^N given in the following table:

R^1	R^2	R^3	R^4
y	y	x	x
z	z	z	y
x	x	y	z

Then $M^2(R^N) = \{(z, y)\}$. We claim that the pair (x, y) is in $C(E^{M^2}, \bar{R}^N)$ according to preference assumption (P2). First, (x, y) is top-ranked for voter 4. Voter 3 only prefers (x, z) or (z, x) to (x, y) but this is not the case for voters 1 and 2. However, voter 3 alone is not effective for $\{(x, z), (z, x)\}$. Finally, voters 1 and 2 only prefer (y, z) and (z, y) to (x, y) , but $\{1, 2\}$ is not effective for $\{(y, z), (z, y)\}$. We conclude that (x, y) is in $C(E^{M^2}, \bar{R}^N)$ under assumption (P2).

Now consider preference assumption (P1). We claim that (x, y) is still in $C(E^{M^2}, \bar{R}^N)$. Voter 4 finds (z, x) and (y, x) better than (x, y) ; also voter 3 finds (z, x) and (y, x) better than (x, y) , but voters 1 and 2 prefer (x, y) over (z, x) and (y, x) , and $\{3, 4\}$ is not effective for $\{(z, x), (y, x)\}$: e.g., $\{1, 2\}$ can put x at bottom so that x can be eliminated first. Voter 3 prefers every alternative in the set $\{(y, x), (z, x), (y, z), (x, z)\}$ to (x, y) , but 1 and 2 prefer only (z, y) over (x, y) , and $\{3\}$ is not effective for $\{(y, x), (z, x), (y, z), (x, z)\}$: $\{1, 2, 4\}$ can put x and z at bottom so that y ends up last. Finally, $\{1, 2\}$ is not effective for $\{(z, y)\}$: e.g., $\{3, 4\}$ can put z at bottom so that z can be eliminated first. We conclude that (x, y) is in $C(E^{M^2}, \bar{R}^N)$ also under assumption (P1).

The fact that M^k satisfies the core property as in [Theorems 3.3 and 3.4](#) means that there is never a coalition of voters which can guarantee an outcome (k -tuple) that is better for all its members in case M^k is used, i.e., if a feasible elimination procedure is used. This fact is certainly not shared by well-known existing methods. The following example illustrates this.

Example 3.6. Let $A = \{a, b, c, d\}$, $N = \{1, 2, 3\}$, and $\beta(x) = 1$ for all $x \in A$. Consider the profile R^N given in the following table:

R^1	R^2	R^3
a	b	c
b	c	a
c	a	b
d	d	d

Suppose we have to select two candidates. In this case, M^2 is quite inconclusive: $M^2(R^N) = \{(x, y) \in \bar{A}^2 : x \neq d, y \neq d\}$. Still, by [Theorems 3.3 and 3.4](#), no coalition of voters can improve on any of the pairs in $M^2(R^N)$. Now consider the method of single transferable vote (STV – see for instance [Brams and Fishburn, 2002](#)). According to STV, first d is eliminated, but thereafter either a or b or c is eliminated. For instance, if a is eliminated, then next b is chosen and finally c , so that in our notation the pair (c, b) results. This way, we obtain $STV(R^N) = \{(c, b), (b, a), (a, c)\}$. Now take, for instance, the pair (b, a) . Both voters 2 and 3 prefer the pair (b, c) over (b, a) – under both (P1) and (P2) – and, moreover, under STV the coalition $\{2, 3\}$ is effective for $\{(b, c)\}$ by the strategy profile

Q^2	Q^3
c	c
b	b
a	a
d	d

In fact, under STV the core for the profile in this example (which is a slightly modified version of the Condorcet paradox) is empty. It is easy to see that the same example applies to methods like plurality voting or run-off elections, assuming that in case of ties we can choose arbitrarily between tied alternatives, as in STV or feasible elimination procedures.

We conclude this section with showing that the SCC M^k is Maskin monotonic. This result will be used in Section 4, but is also of independent interest. A function $H : L^N \rightarrow P_0(\bar{A}^k)$ is *Maskin monotonic* if it satisfies the following. Let $R^N, Q^N \in L^N$ and let $(x_1, \dots, x_k) \in H(Q^N)$. If $x_j Q^i y$ implies $x_j R^i y$ for all $j \in \{1, \dots, k\}$, $y \in A$ and $i \in N$, then $(x_1, \dots, x_k) \in H(R^N)$. For $k = 1$, this definition coincides with the definition of Maskin monotonicity in Section 2.³

Lemma 3.7. M^k is Maskin monotonic.

Proof. Let Q^N and R^N as in the definition of Maskin monotonicity. Without loss of generality we assume that there is a voter v such that $Q^{N \setminus \{v\}} = R^{N \setminus \{v\}}$. Let $f^* = (x_1, C_1; \dots; x_{m-1}, C_{m-1}; x_m)$ be an f.e.p. for Q^N . If $v \notin C_1 \cup \dots \cup C_{m-k}$ then it is easy to see that f^* is still an f.e.p. for R^N , so that $(x_{m-k+1}, \dots, x_m) \in M^k(R^N)$. Now assume $v \in C_1 \cup \dots \cup C_{m-k}$. If $v \in C_j$ with $j > 1$, then we may eliminate x_1, \dots, x_{j-1} and all voters in $C_1 \cup \dots \cup C_{j-1}$ first, and next continue the argument with the remaining profile, where now all voters in C_j have x_j bottom ranked according to Q . So, without loss of generality, let $v \in C_1$.

The rest of the proof is based on a three step algorithm.

Step 1 If the bottom alternative of R^v is equal to x_1 , then f^* is still an f.e.p. for R^N and we are done. Otherwise, go to Step 2.

Step 2 Let the bottom alternative of R^v be $x_\ell \neq x_1$, so $\ell \in \{2, \dots, m-k\}$. If all voters in C_ℓ have x_ℓ as bottom alternative in R^N , then we can first eliminate x_ℓ via C_ℓ and go back to Step 1 for the reduced profile. Otherwise, go to Step 3.

Step 3 Take $\hat{v} \in C_\ell$ with x_ℓ not as bottom alternative and note that the bottom alternative of $R^{\hat{v}} = Q^{\hat{v}}$ is some x_j with $j < \ell$ (since x_j must be eliminated before x_ℓ in f^*). Then modify C_ℓ to $\hat{C}_\ell = (C_\ell \cup \{v\}) \setminus \{\hat{v}\}$ and modify C_1 to $\hat{C}_1 = (C_1 \cup \{\hat{v}\}) \setminus \{v\}$. (In words, we switch v and \hat{v} .) Go back to Step 1.

Repeat this procedure until the final substitute of v in the modified C_1 has x_1 at bottom. Then we can apply an f.e.p. resulting in (x_{m-k+1}, \dots, x_m) , so that $(x_{m-k+1}, \dots, x_m) \in M^k(R^N)$. \square

For $k = 1$, Maskin monotonicity was already proved in Peleg (1984), Lemma 5.3.5.

4. A method for computing M^k in polynomial time

In this section we show that the SCC M^k can be computed in polynomial time. The argument will be based on the following lemma, which, in turn, follows from Maskin monotonicity of M^k . Throughout, M^k is assumed to be defined with respect to weights $\beta(x)$, $x \in A$.

Lemma 4.1. Let $R^N \in L^N$. Then $(x_{m-k+1}, \dots, x_m) \in M^k(R^N)$ if and only if there exist pairwise disjoint subsets $S(y)$, $y \in A \setminus \{x_m\}$, of N such that

- (1) $x_j R^i x_\ell$ for all $\ell = m-k+1, \dots, m-1$, $j > \ell$, and $i \in S(x_\ell)$,
- (2) $x_j R^i y$ for all $y \in A \setminus \{x_{m-k+1}, \dots, x_m\}$, $j = m-k+1, \dots, m$, and $i \in S(y)$,
- (3) $|S(y)| = \beta(y)$ for all $y \in A \setminus \{x_m\}$.

Proof. The only-if direction follows immediately from the definition of an f.e.p. For the if-direction, let $(x_{m-k+1}, \dots, x_m) \in \bar{A}^k$ and assume that there exist subsets $S(y)$ of N , satisfying (1), (2), and (3). List the alternatives of $A \setminus \{x_{m-k+1}, \dots, x_m\}$ as $\{x_1, \dots, x_{m-k}\}$. Consider the following profile $Q^N \in L^N$. For $j = 1, \dots, m-k$ and $i \in S(x_j)$ shift x_j to the bottom of R^i , leaving everything else intact. For all other voters i , let $Q^i = R^i$. Then $(x_1, S(x_1); \dots; x_{m-1}, S(x_{m-1}); x_m)$ is an f.e.p. for Q^N , so that $(x_{m-k+1}, \dots, x_m) \in M^k(Q^N)$. By Maskin monotonicity of M^k (see Lemma 3.7), $(x_{m-k+1}, \dots, x_m) \in M^k(R^N)$. \square

Lemma 4.1 can be used to determine if an alternative (x_{m-k+1}, \dots, x_m) of \bar{A}^k is in $M^k(R^N)$. We first collect a few graph-theoretical concepts in the following definition.

Definition 4.2. A *bipartite graph* is a triple (X, Y, W) , where X and Y are finite disjoint sets and $W \subseteq X \times Y$. Here, the elements of X and Y are called vertices and the elements of W edges. A *matching* is a set $V \subseteq W$ such that for all $(x, y), (x, y'), (x', y) \in V$ we have $y = y'$ and $x = x'$. A matching V is *maximal* if there is no matching V' that has more edges, i.e., there is no matching V' with $|V'| > |V|$.

For $(x_{m-k+1}, \dots, x_m) \in \bar{A}^k$ we define a bipartite graph (X, Y, W) as follows. Let $X = N$ and $Y = \{x_\ell^1, \dots, x_\ell^{\beta(x_\ell)} : \ell = 1, \dots, m-1\}$, where the alternatives in $A \setminus \{x_{m-k+1}, \dots, x_m\}$ are listed as $\{x_1, \dots, x_{m-k}\}$. In other words, the vertices in X

³ Also note that this extended definition of Maskin monotonicity would be implied by Maskin monotonicity in terms of extended preferences according to (P1) or (P2) or any other sensible preference extension.

correspond to the voters and the vertices in Y to the alternatives except x_m , where for each x_ℓ we have $\beta(x_\ell)$ vertices. We define W by

$$W = \{(i, x_\ell^p) \in X \times Y : x_j R^i x_\ell \text{ for all } j \in \{m-k+1, \dots, m\} \text{ with } j > \ell\}.$$

Thus, for every vertex corresponding to an alternative $x_\ell \in A \setminus \{x_{m-k+1}, \dots, x_m\}$, we let there be an edge between this vertex and the vertex corresponding to a voter i if and only if $x_j R^i x_\ell$ for every $j = m-k+1, \dots, m$. For every vertex corresponding to an alternative x_ℓ for some $\ell \in \{m-k+1, \dots, m-1\}$, we let there be an edge between this vertex and the vertex corresponding to a voter i if and only if $x_j R^i x_\ell$ for all $j = \ell+1, \dots, m$.

We now have the following lemma.

Lemma 4.3. *Let $R^N \in L^N$. Then $(x_{m-k+1}, \dots, x_m) \in M^k(R^N)$ if and only if the associated graph (X, Y, W) has a matching V such that for every $x_\ell^p \in Y$ there is an $i \in X$ with $(i, x_\ell^p) \in V$.*

Proof. First, suppose that $(x_{m-k+1}, \dots, x_m) \in M^k(R^N)$. Let the sets $S(x_\ell)$, $\ell = 1, \dots, m-1$ be as in Lemma 4.1. Then for each vertex $x_\ell^p \in Y$ there is an $i_\ell^p \in S(x_\ell) \subseteq X$ such that $(i_\ell^p, x_\ell^p) \in W$ and such that all i_ℓ^p are different. Thus, the set $\{(i_\ell^p, x_\ell^p) : x_\ell^p \in Y\}$ is a matching in (X, Y, W) satisfying the requirement for V in the lemma.

For the converse, suppose that V is a matching in (X, Y, W) as in the lemma. For every $x_\ell \neq x_m$ let $S(x_\ell) = \{i \in X : (i, x_\ell^p) \in V \text{ for some } 1 \leq p \leq \beta(x_\ell)\}$. Then it is easy to verify that these sets $S(x_\ell)$ satisfy the conditions in Lemma 4.1. Hence, $(x_{m-k+1}, \dots, x_m) \in M^k(R^N)$. \square

Note that a matching as in Lemma 4.3 must be maximal, since it contains an edge for every vertex in Y . Thus, by this lemma, in order to decide if $(x_{m-k+1}, \dots, x_m) \in M^k(R^N)$ it is sufficient to find a maximal matching of the associated bipartite graph (X, Y, W) : then $(x_{m-k+1}, \dots, x_m) \in M^k(R^N)$ if and only if the found maximal matching uses all vertices of Y . The problem of finding a maximal matching is polynomial (see Hopcroft and Karp, 1973). Repeating the foregoing procedure $m(m-1) \cdots (m-k+1)$ times is still polynomial (in m and n).⁴

5. Concluding remarks

We conclude with a few brief discussions.

5.1. Other preference extensions

We have shown in this paper that the method for choosing k out of m , based on feasible elimination procedures, has the core property for two intuitive preferences extensions (Theorems 3.3 and 3.4). Unfortunately, as the following example shows, this result does not extend to all reasonable preference extensions.

Example 5.1. Let $A = \{w, x, y, z\}$, $N = \{1, \dots, 5\}$, $\beta(w) = \beta(x) = 1$, and $\beta(y) = \beta(z) = 2$. Consider the profile R^N given in the following table:

R^1	R^2	R^3	R^4	R^5
y	y	z	z	x
w	w	w	w	y
x	x	x	x	z
z	z	y	y	w

Then $(w, x) \in M^2(R^N)$ by the f.e.p. $(z, \{1, 2\}; y, \{3, 4\}; w, \{5\}; x)$. We show that $(w, x) \notin C(E^{M^2}, \bar{R}^N)$ where, for each $i \in N$, \bar{R}^i is the lexicographic preference extension obtained by first comparing best alternatives and then second best alternatives. Consider the following profile for the coalition $\{1, \dots, 4\}$:

Q^1	Q^2	Q^3	Q^4
y	y	y	y
z	z	z	z
x	x	x	x
w	w	w	w

⁴ We thank Ilan Nehama of the Center for the Study of Rationality, Jerusalem, for helpful comments on this subject.

Then $M^2(Q^{\{1,\dots,4\}}, \tilde{R}^5) = \{(z, y)\}$ for all $\tilde{R}^5 \in L$, which implies that $\{(z, y)\} \in E^{M^2}(\{1, \dots, 4\})$. Since $(z, y) \bar{R}^i(w, x)$ for all $i \in \{1, \dots, 4\}$, we conclude that $(w, x) \notin C(E^{M^2}, \bar{R}^N)$.

Observe that other methods do not perform better for the ‘lexicographic best extension’ in this example, by considering again [Example 3.6](#).

5.2. Neutrality

An SCC is *neutral* if it is covariant under permutations of the alternatives. To obtain neutrality of M or, more generally, M^k one needs that all weights $\beta(x)$ are equal, but this is not always possible, due to the restriction on $\beta(\cdot)$. Of course, one can always choose the weights such that the difference between any two weights is at most one. Also, if the number of voters is large relative to the number of alternatives, then one practically obtains neutrality: e.g., if $m = 10$ and $n = 1000$ then one can choose nine of the weights equal to 100 and one weight equal to 101.

Alternatively, one could take the union of the sets $M_\beta^k(R^N)$ over all possible weight assignments $\beta(\cdot)$. It is not clear, however, whether this enlarged neutral SCC still has the core property.

5.3. Further literature

[Peleg and Peters \(2016\)](#) provide an axiomatic characterization of the correspondence M . A characterization of M^k is not yet available.

Among other recent works that are concerned with choosing committees we mention [Kamwa \(2014\)](#) and [Laslier and Van der Straeten \(2014\)](#). The former concentrates on pairwise majority relations. The latter uses approval methods; its Proposition 10 for the case of single-peaked preference profiles coincides with what feasible elimination procedures would achieve in this case.

References

- Abdou, J., Keiding, H., 1991. Effectivity Functions in Social Choice. Kluwer Academic Publishers, Dordrecht.
- Brams, S.J., Fishburn, P.C., 2002. Voting Procedures. Handbook of Social Choice and Welfare, vol. 1. Elsevier, Amsterdam.
- Danilov, V.I., Sotskov, A., 2002. Social Choice Mechanisms. Springer Verlag, Heidelberg.
- Dutta, B., Pattanaik, P.K., 1978. On nicely consistent voting systems. *Econometrica* 46, 163–170.
- Gibbard, A., 1973. Manipulation of voting schemes: a general result. *Econometrica* 41, 587–602.
- Holzman, R., 1986. On strong representations of games by social choice functions. *J. Math. Econ.* 15, 39–57.
- Hopcroft, J.E., Karp, R.M., 1973. An $n^{5/2}$ algorithm for maximum matchings in bipartite graphs. *SIAM J. Comput.* 2, 225–231.
- Ishikawa, S., Nakamura, K., 1980. Representations of characteristic function games by social choice functions. *Int. J. Game Theory* 9, 191–199.
- Kamwa, E., 2014. Essais sur les modes de scrutin et la sélection des comités. PhD thesis. Université de Caen Basse-Normandie.
- Kim, K.H., Roush, F.W., 1981. Properties of consistent voting systems. *Int. J. Game Theory* 10, 45–52.
- Laslier, J.F., Van der Straeten, K., 2014. Strategic Voting Under Committee Approval: A Theory. Toulouse School of Economics and CNRS, Paris.
- Maskin, E., 1999. Nash equilibrium and welfare optimality. *Rev. Econ. Stud.* 66, 23–38.
- Oren, I., 1981. The structure of exactly strongly consistent social choice functions. *J. Math. Econ.* 8, 207–220.
- Peleg, B., 1978. Consistent voting systems. *Econometrica* 46, 153–161.
- Peleg, B., 1984. Game Theoretic Analysis of Voting in Committees. Cambridge University Press, Cambridge.
- Peleg, B., Peters, H., 2006. Consistent voting systems with a continuum of voters. *Soc. Choice Welfare* 27, 477–492.
- Peleg, B., Peters, H., 2010. Strategic Social Choice. Springer, Berlin.
- Peleg, B., Peters, H., 2016. Feasible elimination procedures in social choice: an axiomatic characterization. Maastricht. GSBE Research Memorandum 16/001.
- Polishchuk, I., 1978. Monotonicity and uniqueness of consistent voting systems. Center for Research in Mathematical Economics and Game Theory, Hebrew University of Jerusalem.
- Satterthwaite, M.A., 1975. Strategy-proofness and arrow’s conditions: existence and correspondence theorems for voting procedures and social welfare functions. *J. Econ. Theory* 10, 187–207.